A REMARK ON ZAK'S THEOREM ON TANGENCIES

JOSÉ CARLOS SIERRA

Abstract. We present a slightly different formulation of Zak's theorem on tangencies as well as some applications. In particular, we obtain a better bound on the dimension of the dual variety of a manifold and we classify extremal and next-to-extremal cases when its secant variety does not fill up the ambient projective space.

1. Introduction

In this note, we state Zak's theorem on tangencies ([15, Theorem 0], see also [16, Ch. I, Corollary 1.8]) for non-singular complex varieties in the following way:

Theorem 1 (Reformulation of Zak's theorem on tangencies)**.** *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n, and let $L \subset \mathbb{P}^N$ be a linear subspace of dimension m which is tangent to X along a closed subvariety $Y \subset X$ of dimension r. Then:*

$$r \leq \min\{m - n, \dim SX - 1 - n\}$$

The former statement appears to have some advantages. The new inequality $r \leq \dim SX - 1 - n$ is vacuous if the secant variety of X (denoted by SX) fills up $\mathbb{P}^N$, but it is significant when $SX \neq \mathbb{P}^N$. For instance, this is always the case if $m = N - 1$. In this setting, Zak's theorem on tangencies has several consequences (see [10] for an account) that can be sharpened thanks to Theorem 1. Let $s := \dim SX$ and $c := N - s$.

Corollary 1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n, and let $X^* \subset \mathbb{P}^{N*}$ denote its dual variety (of dimension n^*). The following holds:*

- (i) *The twisted normal bundle $N_{X/\mathbb{P}^N}(-1)$ is k-ample for $k \geq s - 1 - n$ (cf.* [10, Example 6.3.7]*).*
- (ii) *$n^* \geq n + c$ (cf.* [10, Corollary 3.4.20]*). In particular, if $SX \neq \mathbb{P}^N$ then X^* is a singular variety.*
- (iii) *If $s \leq 2n - 1$ (resp. $2n - 2$) then every hyperplane section of X is reduced (resp. normal) (cf.* [10, Corollary 3.4.19]*).*

Some examples of manifolds satisfying the equality $n^* = n$ (and hence $SX = \mathbb{P}^N$) are given by hypersurfaces in $\mathbb{P}^{n+1}$, Segre embeddings $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$, the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$ and the 10-dimensional spinor variety $S_4 \subset \mathbb{P}^{15}$. Moreover, these are the only examples under the additional assumption $3n \leq 2N$ by [2, Theorem 4.5] (cf. Remark 3). On the other hand, if $SX \neq \mathbb{P}^N$ we show that furthermore $n^* \geq n + c + 1$ and manifolds satisfying the equality are classified, giving a new characterization of the Veronese surface:

Theorem 2. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n. If $SX \neq \mathbb{P}^N$ then $n^* \geq n + c + 1$, with equality if and only if X is either a curve or the Veronese surface in $\mathbb{P}^5$.*

Research supported by the "Ramón y Cajal" contract RYC-2009-04999 and the project MTM2009-06964 of MICINN.

Going one step further, let us consider the next-to-extremal case when $SX \neq \mathbb{P}^N$. If $n \leq 3$ it is easy to see (cf. Remark 2) that $n^* = n + c + 2$ if and only if X is either a surface, or a dual defective threefold (i.e. a scroll over a curve), or a secant defective threefold (see [3] for the classification). On the other hand, for $n \geq 4$ we get the following:

Theorem 3. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension $n \geq 4$. If $SX \neq \mathbb{P}^N$ and $n^* = n+c+2$, then X is a scroll over a manifold W and* $\dim W \leq 2$.

If $\dim W = 2$ we will prove in Theorem 6 that the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{n-2}$ is the only scroll as in Theorem 3, so we actually get the following refinement:

Theorem 4. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension $n \geq 4$ such that $SX \neq \mathbb{P}^N$. Then $n^* = n+c+2$ if and only if X is either a scroll over a curve, or (an isomorphic projection of) the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$.*

2. Proofs

Theorem 1 is a consequence of the following application of the Fulton-Hansen connectedness theorem [5]. First, we recall the definition of the relative tangent (resp. secant) variety. Given a subvariety $Y \subset X$, we define $T(Y,X) := \cup_{y \in Y} T_y X$, where $T_y X \subset \mathbb{P}^N$ denotes the embedded tangent space to X at $y \in Y$, and $S(Y,X) \subset \mathbb{P}^N$ as the closure of $\{z \in \mathbb{P}^N \mid \exists y \in Y, \exists x \in X \text{ with } z \in \langle y, x \rangle\}$.

Theorem 5. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n, and let $Y \subset X$ be a closed subvariety of dimension r. Then either* $\dim T(Y,X) = r + n$ *and* $\dim S(Y,X) = r + n + 1$*, or else* $T(Y,X) = S(Y,X)$.

Proof. See [16, Ch. I, Theorem 1.4]. □

We can now prove our results:

Proof of Theorem 1. Let $L \subset \mathbb{P}^N$ be a linear subspace of dimension m which is tangent to X along Y. Then $T(Y,X) \subset L$, but $S(Y,X) \not\subset L$ as $X \subset \mathbb{P}^N$ is non-degenerate. Therefore $T(Y,X) \neq S(Y,X)$, and hence $r + n = \dim T(Y,X) \leq \dim L = m$ by Theorem 5. But Theorem 5 also yields $r + n + 1 = \dim S(Y,X) \leq \dim SX$, so $r \leq \min\{m - n, \dim SX - 1 - n\}$. □

Remark 1. If $SX \neq \mathbb{P}^N$ and $m = N - 1$, the case we are more interested in, the new bound $r \leq s - 1 - n$ is sharp. For example, equality holds for Severi varieties [16, Ch. IV] when $Y \subset X$ is an $n/2$-dimensional quadric.

Proof of Corollary 1. According to Theorem 1, the dimension of the fibres of the second projection of the conormal variety $\mathcal{P}_X := \{(x,H) \mid T_x X \subset H\} \subset X \times \mathbb{P}^{N^*}$ is bounded by $s - 1 - n$. So $N_{X/\mathbb{P}^N}(-1)$ is k-ample for $k \geq s - 1 - n$. This proves (i). Since $\dim \mathcal{P}_X = N - 1$, we get $n^* \geq n + c$. Assume now that X^* is smooth. Then $\dim(X^*)^* \geq n^* \geq n + c$, and Segre's reflexivity theorem $(X^*)^* = X$ ([13], see also [9] for a detailed account) yields $c = 0$, whence $SX = \mathbb{P}^N$ proving (ii). Part (iii) is an immediate consequence of Theorem 1. □

The main ingredients of the proof of Theorem 2 are Zak's classification of Severi varieties and Ein's bound on the defect of subcanonical manifolds [2, Theorem 4.4]:

Proof of Theorem 2. Assume $n \geq 2$ and $n^* \leq n+c+1$. Let $\operatorname{def}(X)$ and $\delta(X)$ denote the dual and secant defect of $X \subset \mathbb{P}^N$, respectively. As $\operatorname{def}(X) := N - 1 - n^*$ and $\delta(X) := 2n + 1 - s$, the inequality $n^* \leq n + c + 1$ is equivalent to the inequality $\operatorname{def}(X) + \delta(X) \geq n - 1$. Since we assume $SX \neq \mathbb{P}^N$, we get $\delta(X) \leq n/2$ by Zak's theorem on linear normality [16, Ch. II, Corollary 2.11] and equality holds if and

only if $X \subset \mathbb{P}^N$ is a Severi variety. Assume first that the Picard group of X is cyclic. Then $\operatorname{def}(X) \leq (n-2)/2$ by [2, Theorem 4.4], and hence

$$n-1 \leq \operatorname{def}(X) + \delta(X) \leq \frac{n-2}{2} + \frac{n}{2} = n-1$$

implies that $X \subset \mathbb{P}^N$ is a Severi variety with $\operatorname{def}(X) = (n-2)/2$. So X is the Veronese surface in $\mathbb{P}^5$. Assume now that the Picard group of X is not cyclic. Then $\delta(X) \leq 2$ by the Barth-Larsen theorem, as otherwise $X \subset \mathbb{P}^N$ could be isomorphically projected into $\mathbb{P}^{2n-2}$ (see for instance [10, Corollary 3.2.3]). Therefore $\operatorname{def}(X) \geq n-3$, whence $\operatorname{def}(X) = n-2$ by Landman's parity theorem (unpublished, see [2, Theorem 2.4]) and $X \subset \mathbb{P}^N$ is a scroll over a curve by [2, Theorem 3.2]. This yields $1 \leq \delta(X) \leq 2$, contradicting Lemma 2. $\square$

Remark 2. (i) The bound $n^* \geq n + c$ given in Theorem 1 is equivalent to the bound $\operatorname{def}(X) + \delta(X) \leq n$. Furthermore, if $SX \neq \mathbb{P}^N$ then $\operatorname{def}(X) + \delta(X) \leq n-1$ by Theorem 2. To the best of the author's knowledge, these relations involving both dual and secant defects appear to be new.

(ii) For $n \geq 3$ the bound obtained in Theorem 2 is equivalent to the bound $\operatorname{def}(X) + \delta(X) \leq n-2$. This can be seen as a refinement of the inequality $\operatorname{def}(X) \leq n-2$ of Landman and Zak when $SX \neq \mathbb{P}^N$ (cf. [16, Ch. I, Remark 2.7]).

(iii) Besides curves and the Veronese surface, the bound $n^* \geq n + c + 2$ (or equivalently $\operatorname{def}(X) + \delta(X) \leq n-2$) is sharp. Equality holds for surfaces with $\delta(X) = 0$, threefolds with $\delta(X) = 1$, scrolls over curves with $\delta(X) = 0$ and the Segre embeddings $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$ with $n \geq 4$. We will prove in the sequel that these are actually the only ones.

The key of the proof of Theorem 3 is a recent characterization of scrolls among dual defective manifolds obtained by Ionescu and Russo in [6]:

Proof of Theorem 3. Let $n^* = n + c + 2$, that is, $\operatorname{def}(X) + \delta(X) = n-2$. If X is a scroll over a manifold W then $\delta(X) \leq 2$ by the Barth-Larsen theorem. Thus $n - 4 \leq \operatorname{def}(X) = n - 2\dim W$, so $\dim W \leq 2$. If X is not a scroll, we can assume $\operatorname{def}(X) \leq (n+1)/3$ by [6, Corollary 3.7] and $\delta(X) \leq (n-1)/2$ by the classification of Severi varieties. Therefore,

$$n-2 = \operatorname{def}(X) + \delta(X) \leq \frac{n+1}{3} + \frac{n-1}{2}$$

yields $n \leq 11$ and, in view of Landman's parity theorem, we get $(n, \operatorname{def}(X), \delta(X)) \in \{(5,1,2), (6,2,2), (9,3,4)\}$. The first two cases are excluded by [1, Theorems 5.1 and 5.2]. In the third case, X is a Fano manifold of dimension 9 with cyclic Picard group generated by the hyperplane section and index $(n + \operatorname{def}(X) + 2)/2 = 7$ (see [2, Lemma 4.2]), so it is ruled out by Mukai's classification of Fano manifolds of coindex 3 (see [11]). $\square$

Remark 3. (i) In a similar way, we can prove that a non-degenerate n-fold $X \subset \mathbb{P}^N$ with $n^* = n$ is either the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$, or else $4n + 5 \geq 3N$ and equality holds if and only if X is the 10-dimensional spinor variety $S_4 \subset \mathbb{P}^{15}$. Since $n^* = n$ we deduce $SX = \mathbb{P}^N$ by Corollary 1, and hence $\delta(X) = 2n + 1 - N$. We point out that $n^* = n$ is equivalent to $\operatorname{def}(X) + \delta(X) = n$. If X is a scroll then $\delta(X) \leq 2$, whence $\operatorname{def}(X) \geq n-2$. Therefore $\operatorname{def}(X) = n-2$, $\delta(X) = 2$ and X is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ by Proposition 1. On the other hand, if X is not a scroll then $\operatorname{def}(X) \leq (n+2)/3$ and equality holds if and only if X is the 10-dimensional spinor variety by [6, Corollary 3.7]. Thus $\delta(X) \geq (2n-2)/3$, and hence $4n + 5 \geq 3N$, with equality if and only if X is the 10-dimensional spinor variety $S_4 \subset \mathbb{P}^{15}$.

(ii) If $n^* = n$ and one furthermore assumes that $3n \leq 2N$ (cf. [2, Theorem 4.5]) then one also gets hypersurfaces and the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$, as in [6, Corollary 3.9].

Proof of Theorem 4. In Theorem 3, if $\dim W = 2$ then $\operatorname{def}(X) = n - 4$ and hence $\delta(X) = 2$, since $\operatorname{def}(X) + \delta(X) = n - 2$. So we conclude in view of Theorem 6. $\square$

3. A result on secant defective scrolls over surfaces

In this section we prove the results on scrolls quoted in Section 2. We say that $X_W \subset \mathbb{P}^N$ (or simply X) is a *scroll* if there exists a vector bundle $\mathcal{E}$ over a manifold W such that $X_W \cong \mathbb{P}_W(\mathcal{E})$ and the fibres of the map $\pi : X_W \to W$, that we denote by F_w for $w \in W$, are linearly embedded in $\mathbb{P}^N$. An equivalent definition of scroll is the following. Let $\mathbb{G}(k,N)$ denote the Grassmannian of k-planes in $\mathbb{P}^N$. Consider the incidence correspondence $\mathcal{U} := \{(\mathbb{P}^k, p) \mid p \in \mathbb{P}^k\}$ with projection maps $\pi_1 : \mathcal{U} \to \mathbb{G}(k,N)$ and $\pi_2 : \mathcal{U} \to \mathbb{P}^N$. For every subvariety $W \subset \mathbb{G}(k,N)$, we denote $\mathcal{U}_W := \pi_1^{-1}(W)$ and $X_W := \pi_2(\mathcal{U}_W)$. Then $X_W \subset \mathbb{P}^N$ is a scroll if and only if W is smooth and $\pi_2 : \mathcal{U}_W \to X_W$ is an isomorphism. The following consequence of Terracini's lemma [14] will be useful. Let $\Sigma_z \subset X$ denote the *entry locus* of $z \in SX$, that is, the closure of the set $\{x \in X \mid \exists x' \in X \text{ with } z \in \langle x, x' \rangle\}$. We recall that $\dim(\Sigma_z) = \delta(X)$ for general $z \in SX$.

Lemma 1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate scroll over W and let $z \in SX$ be a smooth point. If $\Sigma_z \cap F_w \neq \emptyset$ for every $w \in W$ then $SX = \mathbb{P}^N$.*

Proof. Let $T_zSX \subset \mathbb{P}^N$ be the embedded tangent space to SX at z. For every $w \in W$ there exists some $x \in \Sigma_z \cap F_w$, so $F_w \subset T_xX$. Then it follows from Terracini's lemma that $X = \cup_{w \in W} F_w \subset \cup_{x \in \Sigma_z} T_xX \subset T_zSX$. Since $X \subset \mathbb{P}^N$ is non-degenerate we deduce $T_zSX = \mathbb{P}^N$, and hence $SX = \mathbb{P}^N$. $\square$

The following lemma is well known. We include a short proof based on Lemma 1:

Lemma 2. *Let $X \subset \mathbb{P}^N$ be a non-degenerate scroll over a curve. If $\delta(X) > 0$ then $SX = \mathbb{P}^N$.*

Proof. Since $\dim \Sigma_z = \delta(X) > 0$ for general $z \in SX$ and $\dim W = 1$, we deduce that $\Sigma_z \cap F_w \neq \emptyset$ for every $w \in W$. Therefore $SX = \mathbb{P}^N$ by Lemma 1. $\square$

Let $X_W \subset \mathbb{P}^N$ be an n-dimensional scroll. It follows from the Barth-Larsen theorem that $\delta(X) \leq 2$. From now on, we will focus on the extremal case $\delta(X) = 2$. On the one hand, if W is a curve then $X_W \subset \mathbb{P}^N$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ (see [8, pp. 307–308]). We prove this result in a more geometric and elementary way. The idea of the proof is essentially due to Fyodor Zak:

Proposition 1. *The only n-dimensional scroll over a curve in $\mathbb{P}^{2n-1}$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$.*

Proof. For every $w \in W$, let $\sigma_w := \{g \in \mathbb{G}(n-1, 2n-1) \mid \mathbb{P}^{n-1}_g \cap F_w \neq \emptyset\}$. Then σ_w is a hyperplane section of $\mathbb{G}(n-1, 2n-1)$, embedded by Plücker, and $w \in \sigma_w$ is a singular point of multiplicity n. The intersection of W and σ_w is supported at w since $\pi_2 : \mathcal{U}_W \to X_W$ is injective. Moreover, W and σ_w meet transversally at w since $d\pi_2$ is injective. So we deduce that $\deg X_W = \deg W = m_w(\sigma_w) \cdot m_w(W) = n$, where $m_w(\sigma_w)$ and $m_w(W)$ denote the multiplicity of σ_w and W at w, respectively (see [4, Corollary 12.4]). Therefore, for every $w \in W$ there exists a hyperplane section σ_w in the Plücker embedding of W such that the intersection product $\sigma_w \cdot W = nw$. This property characterizes the rational normal curve of degree n, so $X_W \subset \mathbb{P}^{2n-1}$ is a non-degenerate (otherwise $F_w \cap F_{w'} \neq \emptyset$ for every $w' \in W$) rational normal scroll of degree n. Consequently, X_W is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$. $\square$

Remark 4. The hypothesis of Proposition 1 can be weakened. Arguing with a general $w \in W$, the same proof works if $W \subset \mathbb{G}(n-1, 2n-1)$ is an integral curve and $\pi_2 : \mathcal{U}_W \to X_W$ is an isomorphism (or even if $\pi_2 : \mathcal{U}_W \to X_W$ has finitely many double points).

On the other hand, if W is a surface there exists a complete classification of scrolls with $\delta(X) = 2$ only for $n = 3$ (see [12] and [7, Proposition 4]). We now prove that (an isomorphic projection of) the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$ with $n \geq 4$ is the only scroll over a surface whose secant variety does not fill up the ambient space. The main idea of the proof is to show that $X \subset \mathbb{P}^N$ is swept out by a 2-dimensional family of Segre embeddings $\mathbb{P}^1 \times \mathbb{P}^{n-2}$. More precisely, we prove that any two fibres of the scroll, F_w and $F_{w'}$, determine a Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ in the linear span $\langle F_w, F_{w'} \rangle =: \mathbb{P}^{2n-3}_{ww'}$ that they define.

Theorem 6. *Let $X \subset \mathbb{P}^N$ be a non-degenerate scroll of dimension n over a surface. If $\delta(X) = 2$ and $SX \neq \mathbb{P}^N$ then X is (an isomorphic projection of) the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$.*

Proof. We claim that $\dim S(F_w, X) = 2n-2$ for general $w \in W$. Since $\langle F_w, F_{w'} \rangle = S(F_w, F_{w'}) \subsetneq S(F_w, X)$ for every $w' \in W$, we get $\dim S(F_w, X) \geq 2n-2$. Fix a general $z \in SX$. We deduce from Lemma 1 that $\Sigma_z \cap F_w = \emptyset$ for general $w \in W$, and hence $z \notin S(F_w, X)$. Therefore $S(F_w, X) \subsetneq SX$ proving the claim, as $\dim SX = 2n-1$. For every $w \in W$, consider the subvariety $\mathcal{G}_w := \{\langle F_w, F_{w'} \rangle \mid w' \in W\} \subset \mathbb{G}(2n-3, N)$. If $\dim \mathcal{G}_w = 2$ for general $w \in W$ then $S(F_w, X) \subset \mathbb{P}^N$ is a $(2n-2)$-dimensional subvariety swept out by a 2-dimensional family of $(2n-3)$-dimensional linear subspaces, so $S(F_w, X) \subsetneq \mathbb{P}^N$ itself is a linear subspace. This contradicts the non-degeneracy of $X \subset \mathbb{P}^N$. Thus $\dim \mathcal{G}_w = 1$ for general (and hence every) $w \in W$. In particular, for every $w, w' \in W$ there exists an integral curve $T_{ww'} \subset W$ such that $\langle F_w, F_{w'} \rangle = \langle F_w, F_{w''} \rangle$ for every $w'' \in T_{ww'}$. So $X_{T_{ww'}} = \mathbb{P}^1 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{2n-3}_{ww'}$ by Remark 4. Consequently, for every $w \in W$ and every $x \in F_w$ there exists a 1-dimensional family of lines each of them meeting F_w at x and giving a 2-dimensional cone $C_x \subset X$. Since F_w and C_x are contained in $T_xX = \mathbb{P}^n$ we deduce $\deg(C_x) = C_x \cdot F_w$. We claim that $C_x \cdot F_{w'} = 1$ for every $w' \in W$, and hence $\deg(C_x) = C_x \cdot F_w = C_x \cdot F_{w'} = 1$. Let us prove the claim. If $C_x \cdot F_{w'} \geq 2$ then $T_xX = \langle F_w, C_x \cap F_{w'} \rangle \subset \langle F_w, F_{w'} \rangle$. Therefore $T(F_w, X) \subset \langle F_w, F_{w'} \rangle$, contradicting Theorem 1. Since $\deg(C_x) = C_x \cdot F_w = 1$, we deduce that $C_x = \mathbb{P}^2$ for every $x \in F_w$ and that C_x is a section of $\pi : X_W \to W$ (in particular, $W \cong \mathbb{P}^2$). Thus X is a scroll over $\mathbb{P}^2$ and $\mathbb{P}^{2n-2}$, respectively. So $X \subset \mathbb{P}^N$ is an isomorphic projection of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$. □

Acknowledgements

The author is grateful to Fyodor Zak for helpful comments and encouragement.

References

[1] L. Ein, *Varieties with small dual varieties II*, Duke Math. J. **52** (1985), 895–907.
[2] ———, *Varieties with small dual varieties I*, Invent. Math. **86** (1986), 63–74.
[3] T. Fujita, *Projective threefolds with small secant varieties*, Sci. Papers College Gen. Ed. Univ. Tokyo **32** (1982), 33–46.
[4] W. Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.
[5] W. Fulton and J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. (2) **110** (1979), 159–166.
[6] P. Ionescu and F. Russo, *Manifolds covered by lines, defective manifolds and a restricted hartshorne conjecture*, arXiv:0909.2763v2 [math.AG].

[7] P. Ionescu and M. Toma, *Boundedness for some special families of embedded manifolds*, Classification of algebraic varieties (L'Aquila, 1992), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 215–225.
[8] S. L. Kleiman, *Plane forms and multiple-point formulas*, Algebraic threefolds (Varenna, 1981), Lecture Notes in Math., vol. 947, Springer, Berlin, 1982, pp. 287–310.
[9] ———, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, pp. 163–225.
[10] R. Lazarsfeld, *Positivity in algebraic geometry I–II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48–49, Springer-Verlag, Berlin, 2004.
[11] S. Mukai, *Biregular classification of fano* 3*-folds and fano manifolds of coindex* 3, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), 3000–3002.
[12] G. Ottaviani, *On* 3*-folds in* $\mathbf{p}^5$ *which are scrolls*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), 451–471.
[13] C. Segre, *Preliminari di una teoria delle varietà luoghi di spazi*, Rend. Circ. Mat. Palermo **30** (1910), 87–121.
[14] A. Terracini, *Sulle* V_k *per cui la varietà degli* S_h *(h+1)-seganti ha dimensione minore dell' ordinario*, Rend. Circ. Mat. Palermo **31** (1911), 392–396.
[15] F. L. Zak, *Projections of algebraic varieties*, Mat. Sb. (N.S.) **116(158)** (1981), 593–602, English translation in Mat. USSR Sb. **44** (1983) 535–544.
[16] ———, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993.

Instituto de Ciencias Matemáticas (ICMAT), Consejo Superior de Investigaciones Científicas (CSIC), Campus de Cantoblanco, 28049 Madrid, Spain
E-mail address: jcsierra@icmat.es